Bimodule Properties of Group-Valued Local Fields and Quantum-Group Difference Equations

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Ling-Lie Chau and Itaru Yamanaka

Physics Department, University of California, Davis, CA 95616

Abstract

We give an explicit construction of the quantum-group generators — local, semi-local, and global — in terms of the group-valued quantum fields $\tilde{g}$ and $\tilde{g}^{-1}$ in the Wess-Zumino-Novikov-Witten (WZNW) theory. The algebras among the generators and the fields make concrete and clear the bimodule properties of the $\tilde{g}$ and the $\tilde{g}^{-1}$ fields. We show that the correlation functions of the $\tilde{g}$ and $\tilde{g}^{-1}$ fields in the vacuum state defined through the semi-local quantum-group generator satisfy a set of quantum-group difference equations. We give the explicit solution for the two point function. A similar formulation can also be done for the quantum Self-dual Yang-Mills (SDYM) theory in four dimensions.

The Wess-Zumino-Novikov-Witten (WZNW) theory [1, 2] has a long and beautiful history. In his well known 1984 paper Witten [2] quantized the Lie-algebra valued field $\tilde{j}_\mu$ of the theory and derived its current algebra with central charge. From this current algebra Knizhnik and Zamolodchikov [3] derived the linear equations (the K-Z equations) satisfied by the correlation functions. (In their original formulation the Virasoro generators [4] played an important role. Actually one can obtain the K-Z equations without involving the Virasoro-algebra generators [5].) Later the quantum-group structures of the theory were discovered [6, 7] and studied in many papers, [8] to [11]. However, in all of these studies the role of the group-valued local quantum fields $\tilde{g}$, the basic fields in the theory, was not clear.

Recently the WZNW theory was studied from the point of view of considering the fields $\tilde{g}$ as the basic fields, papers [12] to [16]. We quantized the group-valued local quantum fields $\tilde{g}$ of the Wess-Zumino action in the light-cone coordinates [15, 16] using the Dirac procedure for constrained systems [17]. Further, we had also formulated the quantum Self-dual Yang-Mills (SDYM) system [18] in terms of the group-valued local field $\tilde{J}$ and showed how the two theories are related. The quantum WZNW theory in terms of $\tilde{g}$ can be obtained from the quantum SDYM theory in terms of $\tilde{J}$ by reducing the two appropriate dimensions in the quantum SDYM theory. The exchange algebras satisfied by the group-valued local quantum fields in the two theories are very similar. In both cases we showed that the second-class constraints in forming the Dirac brackets in the light-cone coordinates are the source of the nontrivial critical exponents in the products of fields and the quantum-group structures in these theories. However there are very important differences between the two theories. The WZNW theory is a free theory in the light-cone-coordinate formulation in two dimensions. The self-dual Yang-Mills theory is an interacting theory (even in the light-cone coordinate formulation) in four dimensions. One can easily see how the interactions disappear in the dimension reduction. Because of its simpler structure, the quantum WZNW theory is an important laboratory for the study of the quantum SDYM field theory, which in turn is an important laboratory for the study of many other quantum field theories in four dimensions [19].

In addition to the difference in our way of obtaining the exchange algebra from those of Refs. [12] to [14], there are other important differences in our interpretation of the exchange algebra and in the further development of theory. We have given an analytic interpreta-

tion to the spatial dependence of the R matrix of the exchange algebras of the $\tilde{g}$ fields. From that interpretation we have formulated the normal-ordering procedure, constructed the $\tilde{g}^{-1}$ quantum fields and their exchange algebras, constructed the Lie-algebraic current $\tilde{j}(\bar{y}) \sim \tilde{g}\partial_{\bar{y}}\tilde{g}^{-1}$, and derived the current algebra from the exchange algebras of $\tilde{g}$ and $\tilde{g}^{-1}$ without resorting to the use of the boson quantum fields [20]. This procedure also makes it straightforward to construct the theory for the general sl(n) cases.

What has emerged is that the group-valued local quantum fields, $\tilde{g}$ and $\tilde{g}^{-1}$ are bimodule quantum fields. Dictated by the R matrix and the nontrivial critical exponents, the $\tilde{g}$ fields form noncommutative vector spaces of the quantum-group on the right side and commutative vector spaces on the left side. The left side of $\tilde{g}$ form the fundamental representation of the Lie group and the Lie-algebra currents $\tilde{j}$ are the generators for its transformation. The right side of $\tilde{g}$ forms the representation of the quantum-group. However, until now the generators of the quantum-group transformations had not been fully constructed and it was unknown whether or not they could be constructed from $\tilde{g}$ and $\tilde{g}^{-1}$ quantum fields. (The above statements apply similarly to $\tilde{g}^{-1}$, except that the roles of the two sides are reversed.)

In this paper we give an explicit construction of the quantum-group generators — local, semi-local, and global ones — in terms of the quantum $\tilde{g}$ and $\tilde{g}^{-1}$ fields. Their algebras make clear and concrete the bimodule properties of the $\tilde{g}$ and the $\tilde{g}^{-1}$ fields. From the semi-local quantum-group generators, we define a vacuum which we call the $U_q^{\Delta}[\mathrm{sl}(n)]$-vacuum. It is different from the vacuum defined from the regular Lie-algebra current $\tilde{j}(\bar{y})$, which we call the $\widehat{\mathrm{sl}(n)}$-vacuum. The two vacua are different since the semi-local quantum-group generators do not commute with the Lie-algebra current. We then show that the $U_q^{\Delta}[\mathrm{sl}(n)]$-vacuum-expectation-values of products of the $\tilde{g}$ and $\tilde{g}^{-1}$ fields satisfy a set of linear difference equations, which we call the quantum-group difference equations. (These equations look different from those discussed in Ref. [5].) For the two point function we provide the solution. With these additional understandings, the bimodule properties of the WZNW theory become concrete and clear.

The bimodule properties, we believe, are generic for all group-valued local quantum fields as we showed earlier [18] that they also hold in the SDYM quantum field theory. For the interacting SDYM quantum field theory the exchange algebras of the group-valued quantum local fields $\tilde{J}$ and $\tilde{J}^{-1}$are just the starting point of the theory. The exchange

algebras and the current algebras in the SDYM theory are fixed-time relations to begin with. Using one of the additional dimensions available and performing its spatial-integration, in paper [18] we constructed time-independent currents and their current algebras, with interesting features of higher dimensions. From these time-independent currents [21] we are able to construct a time-independent local monodromy matrix $\tilde{T}\circ$ and derived its local exchange algebra $R^T\tilde{T}\circ\tilde{T}\circ = \tilde{T}\circ\tilde{T}\circ R^T$ which contains the infinite local charge algebras and the infinite local Yangian algebras and then derive differential and difference equations for their correlation functions in the SDYM theory [22].

The organization of the rest of the paper is the following. We first give a brief description of our formulation of the quantum WZNW theory. To be specific and simple, we discuss here the case of sl(2). Our formulation can be generalized to the cases of sl(n) in a straightforward way [23]. We then give the construction of the quantum-group generators and their algebras. After defining the U_q^{Δ}[sl(2)]-vacuum through the semi-local quantum-group generators $\tilde{G}^{\Delta}(\bar{y}) \equiv \tilde{g}^{-1}(\bar{y}-\Delta)\ \tilde{g}(\bar{y}+\Delta)$, we show that the correlation functions of the $\tilde{g}$ and $\tilde{g}^{-1}$ fields defined in this vacuum satisfy a set of difference equations. For the two point function we give the explicit solution. We then end the paper with some concluding remarks.

Exchange Algebras, Critical Exponents, Normal-Ordering, and Current Algebras

In this section we briefly describe the essential points of our formulation of the quantum WZNW theory in terms of the $\tilde{g}$ and $\tilde{g}^{-1}$ fields so that the new development can be clearly presented. In the case of sl(2), $\tilde{g}$ is a 2×2 matrix with non-commuting operator-valued entries, which we call the quantum $\tilde{g}$ fields. They satisfy the following exchange algebras,

$$\tilde{g}_I(\bar{y}_1)\tilde{g}_{II}(\bar{y}_2) = 1_{I,II}\tilde{g}_{II}(\bar{y}_2)\tilde{g}_I(\bar{y}_1)R_{I,II}(\bar{y}_1-\bar{y}_2), \tag{1}$$

where

$$R_{I,II} = P_{I,II}\left\{[q]^{\triangle_1\ \varepsilon(\bar{y}_1-\bar{y}_2)}\mathcal{P}^q_{j_{12}=1} - [q]^{\triangle_0\ \varepsilon(\bar{y}_1-\bar{y}_2)}\mathcal{P}^q_{j_{12}=0}\right\}. \tag{2}$$

Here we denote the light-cone coordinate $x-t\equiv\bar{y}$. The time in this light-cone-coordinate formulation is $y\equiv x+t$. In Eq. (2),

$$q \equiv e^{-i\hbar/4\alpha}, \tag{3}$$

which becomes unity when $\hbar \to 0$; α is the coefficient in front of the Wess-Zumino action; $\triangle_1 = 2\frac{1}{2}(\frac{1}{2}+1) - 1(1+1) = -1/2$ and $\triangle_0 = 2\frac{1}{2}(\frac{1}{2}+1) - 0(0+1) = 3/2$ are the differences of conformal dimensions of two spin $\frac{1}{2}$ fields minus that of a spin $j_{12} = 1$ and 0 fields, respectively ; the subscripts I and II denote the tensor spaces that the operator matrices or c-number matrices operate on. (This tensor notation saves us the trouble of writing out the indices of the matrix elements. Written in terms of the matrix elements, Eq. (1) reads $\tilde{g}_{m_1,\alpha_1}(\bar{y}_1)\tilde{g}_{m_2,\alpha_2}(\bar{y}_2) = \delta_{m_1,l_1}\delta_{m_2,l_2}\tilde{g}_{l_2,\beta_2}(\bar{y}_2)\tilde{g}_{l_1,\beta_1}(\bar{y}_1)R_{\beta_1,\beta_2;\alpha_1,\alpha_2}(\bar{y}_1 - \bar{y}_2)$, where the repeated indices l_1, l_2, β_1, and β_2 are summed.) The $\mathcal{P}^q_{j_{12}}$'s are the 4×4 c-number q-ed projection matrices projecting the two spin 1/2 states into $j_{12} = 0$ or 1, satisfying $\mathcal{P}^q_{j_{12}}\mathcal{P}^q_{j'_{12}} = \mathcal{P}^q_{j_{12}}\delta_{j_{12}j'_{12}}$,

$$\mathcal{P}^q_{j_{12}=1} = \mathrm{diag}\{1, a\begin{pmatrix} q & 1 \\ 1 & q^{-1}\end{pmatrix}, 1\}, \quad \mathcal{P}^q_{j_{12}=0} = 1 - \mathcal{P}^q_{j_{12}=1}, \tag{4}$$

where $a \equiv 1/(q+q^{-1}) \equiv 1/[2]_q$, with $[n]_q \equiv (q^n - q^{-n})/(q - q^{-1})$. The matrix $P_{I,II}$ interchanges matrices in space I to II and visa versa, e.g., $P_{I,II}\, \tilde{g}_I(\bar{y}_1)\, \tilde{g}_{II}(\bar{y}_2) = \tilde{g}_{II}(\bar{y}_1)\, \tilde{g}_I(\bar{y}_2)\, P_{I,II}$, and its explicit representation is $P_{I,II} = \frac{1}{2}(1 - \sum_{a=1}^3 \sigma^a\sigma^a) = \mathcal{P}_{j_{12}=1} - \mathcal{P}_{j_{12}=0}$; here the $\mathcal{P}_{j_{12}}$'s are the un-q-ed ordinary projection matrices, i.e., Eq. (4) with $q = 1$.

The q-deformation in the R matrix, i.e., its quantum-group structure, is an $\hbar$ effect, $R = 1$ when $\hbar = 0$. Taking the leading term in $\hbar$, Eq. (1) becomes the Dirac bracket of the g fields. Therefore we directly identified that the second-class constraints are the source of the quantum-group structure of the theory, [15] and [16].

The $\varepsilon(\bar{y}_1 - \bar{y}_2)$ in Eq. (2) is a signature of the quantizaton in the light-cone coordinates. It has the usual definition

$$\varepsilon(\bar{y}_1 - \bar{y}_2) = \pm 1, \quad \text{for } \ \bar{y}_1 \gtrless \bar{y}_2. \tag{5}$$

We denote $R_{I,II}(\bar{y}_1 - \bar{y}_2) = R_{I,II}(+)$, for $\bar{y}_1 - \bar{y}_2 > 0$ and $R_{I,II}(\bar{y}_1 - \bar{y}_2) = R_{I,II}(-)$, for $\bar{y}_1 - \bar{y}_2 < 0$. Note that $[R_{I,II}(+)]^{-1} = R_{II,I}(-)$ and $[R_{I,II}(-)]^{-1} = R_{II,I}(+)$. In our formulation we emphasize the analytic-function interpretation of $\varepsilon(\bar{y}_1 - \bar{y}_2)$, i.e.,

$$\varepsilon(\bar{y}_1 - \bar{y}_2) = -[ln(\bar{y}_1 - \bar{y}_2 + i\varepsilon) - ln(\bar{y}_2 - \bar{y}_1 + i\varepsilon)]/\pi i, \tag{6}$$

and define

$$\varepsilon(\bar{y}_1 - \bar{y}_2) = 0, \quad \text{for } \ \bar{y}_1 = \bar{y}_2. \tag{7}$$

The expression for $\epsilon(\bar{y}_1-\bar{y}_2)$, Eq. (6), indicates that the product $\tilde{g}_I(\bar{y}_1)\tilde{g}_{II}(\bar{y}_2)$ has singularity at $\bar{y}_1-\bar{y}_2=0$ with critical exponents given by

$$\mathcal{P}_{j_{12}}\tilde{g}_I(\bar{y}_1)\tilde{g}_{II}(\bar{y}_2)\mathcal{P}^q_{j'_{12}} = (\bar{y}_1-\bar{y}_2)^{\triangle_{j_{12}}(ln\ q)/\pi i}\{:\mathcal{P}_{j_{12}}\tilde{g}_I(\bar{y}_1)\tilde{g}_{II}(\bar{y}_2)\mathcal{P}^q_{j'_{12}}:\}. \tag{8}$$

This also defines the normal-order products to be those in the curly brackets; their Taylor expansions give the operator-product expansions.

Using Eq. (7), we can easily prove that at $\bar{y}_1=\bar{y}_2$, the exchange algebra Eq. (1) gives

$$\mathcal{P}_{j_{12}}g_I(\bar{y}_1)g_{II}(\bar{y}_1) = g_I(\bar{y}_1)g_{II}(\bar{y}_1)\mathcal{P}^q_{j_{12}}, \tag{9}$$

where $j_{12}=0,1$. Eq. (9) implies $\mathcal{P}_{j_{12}}g_I(\bar{y}_1)g_{II}(\bar{y}_1)\mathcal{P}^q_{j'_{12}}=0$, for $j_{12}\neq j'_{12}$. This fact and the later development of the quantum-group generators rely crucially on the interpretation of the R matrix at the coincidence point, Eq. (7).

The quantum $\tilde{g}$ field can be interpreted as a non-Abelian vertex operator. However, notice that our $\tilde{g}$ field in component form has the bimodule indices. The vertex operators given in Refs. [9], [10], and [11] have one less index and no bimodule property indicated.

We defined the $\tilde{g}^{-1}$ field from the following equation

$$\tilde{g}(\bar{y})\tilde{g}^{-1}(\bar{y}) = 1 = \tilde{g}^{-1}(\bar{y})\tilde{g}(\bar{y}). \tag{10}$$

From Eqs. (10) and (1), we can easily show that the $\tilde{g}^{-1}$ field satisfies the following exchange algebras

$$\tilde{g}_I^{-1}(\bar{y}_1)\tilde{g}_{II}(\bar{y}_2) = \tilde{g}_{II}(\bar{y}_2)[R_{I,II}(\bar{y}_1-\bar{y}_2)]^{-1}\tilde{g}_I^{-1}(\bar{y}_1), \tag{11}$$

and

$$\tilde{g}_I^{-1}(\bar{y}_1)\tilde{g}_{II}^{-1}(\bar{y}_2) = R_{I,II}(\bar{y}_1-\bar{y}_2)\tilde{g}_{II}^{-1}(\bar{y}_2)\tilde{g}_I^{-1}(\bar{y}_1). \tag{12}$$

The construction of this $\tilde{g}^{-1}$ field is crucial for us to develop the full content of the theory in terms of the group-valued fields.

From $\tilde{g}$ and $\tilde{g}^{-1}$, we constructed the $\widehat{\mathrm{sl}(2)}$ current

$$\tilde{j}(\bar{y}) \equiv 2\alpha\ \tilde{g}\partial_{\bar{y}}\tilde{g}^{-1}. \tag{13}$$

We then showed that the following equations can be easily derived from the exchange algebras of the $\tilde{g}$ and $\tilde{g}^{-1}$ quantum fields,

$$[\tilde{j}_I(\bar{y}_1),\tilde{j}_{II}(\bar{y}_2)] = -i\hbar[M_{I,II},\tilde{j}_{II}(\bar{y}_1)]\ \delta(\bar{y}_1-\bar{y}_2) - i\hbar 2\alpha M_{I,II}\ \delta'(\bar{y}_1-\bar{y}_2), \tag{14}$$

$$[\tilde{j}_I(\bar{y}_1),\tilde{g}_{II}(\bar{y}_2)] = -i\hbar M_{I,II}\ \tilde{g}_{II}(\bar{y}_1)\delta(\bar{y}_1-\bar{y}_2), \tag{15}$$

$$[\tilde{j}_I(\bar{y}_1),\tilde{g}_{II}^{-1}(\bar{y}_2)] = i\hbar\tilde{g}_{II}^{-1}(\bar{y}_1)M_{I,II}\ \delta(\bar{y}_1-\bar{y}_2), \tag{16}$$

where $M_{I,II} \equiv P_{I,II} - \frac{1}{2} = \frac{1}{2}\sum_{a=1}^{3} \sigma_I^a \sigma_{II}^a$. Eq. (14) is the current algebra of the current $\tilde{j}$. Taking trace of Eq. (14) onto $\frac{1}{2i}\sigma_I^a$ and $\frac{1}{2i}\sigma_{II}^b$ one can easily obtain the more familiar form of the current algebra in terms of the Lie-components of the current $[\tilde{j}^a(\bar{y}_1), \tilde{j}^b(\bar{y}_2)] = i\hbar\varepsilon^{abc}\tilde{j}^c(\bar{y}_1)\ \delta(\bar{y}_1 - \bar{y}_2) - \frac{i\hbar^2}{4\pi} K\delta^{ab}\ \delta'(\bar{y}_1 - \bar{y}_2)$, where $K \equiv i\pi[ln(q)]^{-1} = -4\pi\alpha/\hbar$. Therefore we had reproduced the well known current algebra given by Witten in Ref. [2], but we have constructed from the group-valued quantum fields $\tilde{g}$ and $\tilde{g}^{-1}$, a different quantum formulation of WZNW theory. Eq. (15) indicates that the left side of $\tilde{g}$ forms the fundamental representation of the current $\tilde{j}$; Eq. (16) indicates that the right side of $\tilde{g}^{-1}$ forms the fundamental representation of the current $\tilde{j}$. From

$$2\pi i\ \delta(\bar{y}_1 - \bar{y}_2) = \frac{1}{\bar{y}_1 - \bar{y}_2 - i\varepsilon} - \frac{1}{\bar{y}_1 - \bar{y}_2 + i\varepsilon}, \tag{17}$$

the δ-function on the right-hand-side of Eqs. (14) to (16) indicats that those products of fields have singularities. Equations Eqs. (14) to (16) can be equivalently written out as operator-product-expansions for products of operators (which we leave as exercises for the reader). Next we present our new development.

Quantum-Group Currents and Global Quantum-Group Generators

Similar to the construction of the current $\tilde{j}(\bar{y})$, it is nature to construct the other current $\tilde{j}^q(\bar{y})$

$$\tilde{j}^q(\bar{y}) \equiv 2\alpha\ \tilde{g}^{-1}(\bar{y})\partial_{\bar{y}}\tilde{g}(\bar{y}), \tag{18}$$

which we shall call the quantum-group current since it has the quantum-group index on both sides. We can work out the algebraic relations among the matrix elements of $\tilde{j}^q(\bar{y})$, corresponding to Eq. (14) for $\tilde{j}(\bar{y})$; and the algebraic relations with the fields $\tilde{g}$ and $\tilde{g}^{-1}$, corresponding to Eqs. (15) and (16). All of them have nice quantum-group interpretations. However, we find that $\tilde{j}^q$ is not as useful a quantity as the current $\tilde{j}$ in that it can not be used to develop its vacuum states and the corresponding differential equations as the current $\tilde{j}$ was used to develop the K-Z equations. Therefore here we do not write out those algebraic relations involving $\tilde{j}^q$.

On the other hand, we find that the following group-valued quantities, $\tilde{G}$ and $\tilde{G}^\Delta$, are the more useful quantum-group generators. We form the global quantum-group generator

$\tilde{G}$ from the quantum-group current $\tilde{j}^q$ of Eq. (18) by a path ordered integration,

$$\tilde{G} = \vec{P}exp(\int_{-\infty}^{\infty} d\bar{y}\ \tilde{g}^{-1}\partial_{\bar{y}}\tilde{g}) = \tilde{g}^{-1}(-\infty)\tilde{g}(\infty) \quad . \tag{19}$$

From the exchange algebras, Eqs. (1), (11), and (12), we can derive the algebraic relations between the matrix elements of $\tilde{G}$ and the action of $\tilde{G}$ on $\tilde{g}$ and $\tilde{g}^{-1}$,

$$\begin{aligned} \{R_{II,I}(+)\ \tilde{G}_I\ R_{I,II}(+)\ \}\tilde{G}_{II} &= \tilde{G}_{II}\ \{R_{II,I}(+)\ \tilde{G}_I\ R_{I,II}(+)\}, & (20)\\ \tilde{G}_I\ \tilde{g}_{II} &= \tilde{g}_{II}\ \{R_{II,I}(+)\ \tilde{G}_I\ R_{I,II}(+)\}, & (21)\\ \tilde{g}_{II}^{-1}\ \tilde{G}_I &= \{R_{II,I}(+)\ \tilde{G}_I\ R_{I,II}(+)\}\ \tilde{g}_{II}^{-1}, & (22) \end{aligned}$$

where we use the curly bracket to bracket relevant quantities together to make the meaning of equations clearler. Associativity for the products of the fields are true because the R matrix satisfies the Yang-Baxter relations (which we leave as an excercise for the reader.) Eqs. (20) to (22) are the algebraic relations for $\tilde{G}$ parallel to those of Eqs. (14) to (16) for $\tilde{j}$.

The basic elements of the quantum-group generators $\{\tilde{e}_i; i = 3,\ \text{and}\ \pm\}$ are related to the components of the matrix $\tilde{G}$ by

$$\tilde{G} \equiv \begin{pmatrix} 1 & 0 \\ (1-q^2)\ \tilde{e}_+ & 1 \end{pmatrix} \begin{pmatrix} q^{-\tilde{e}_3} & 0 \\ 0 & q^{\tilde{e}_3} \end{pmatrix} \begin{pmatrix} 1 & (q^{-1}-q)\ \tilde{e}_- \\ 0 & 1 \end{pmatrix}, \tag{23}$$

where $\tilde{e}_\pm$ and $q^{-\tilde{e}_3}$ satisfy the standard quantum-groups algebras [7, 24] which guarantee Eqs. (20) to (22).

Semi-local Quantum-Group Generators

Changing the integration range of Eq. (19) to a finite one, we obtain the semi-local quantum-group generators

$$\tilde{G}^\Delta(\bar{y}) \equiv \vec{P}exp(\int_{\bar{y}-\Delta}^{\bar{y}+\Delta} d\bar{y}'\ \tilde{g}^{-1}\partial_{\bar{y}'}\tilde{g}) = \tilde{g}^{-1}(\bar{y}-\Delta)\ \tilde{g}(\bar{y}+\Delta). \tag{24}$$

Again using the exchange algebras, Eqs. (1), (11), and (12), we obtain

$$\begin{aligned} &\{R_{I,II}^{-1}(\bar{y}_1-\bar{y}_2)\ \tilde{G}_I^\Delta(\bar{y}_1)\ R_{I,II}(\bar{y}_1-\bar{y}_2+2\Delta)\}\ \tilde{G}_{II}^\Delta(\bar{y}_2) \\ &\quad = \{\tilde{G}_{II}^\Delta(\bar{y}_2)\ R_{I,II}^{-1}(\bar{y}_1-\bar{y}_2-2\Delta)\ \tilde{G}_I^\Delta(\bar{y}_1)\}\ R_{I,II}(\bar{y}_1-\bar{y}_2), & (25)\\ \tilde{G}_I^\Delta(\bar{y}_1)\ \tilde{g}_{II}(\bar{y}_2) &= \tilde{g}_{II}(\bar{y}_2)\ \{R_{I,II}^{-1}(\bar{y}_1-\bar{y}_2-\Delta)\ \tilde{G}_I^\Delta(\bar{y}_1)\ R_{I,II}(\bar{y}_1-\bar{y}_2+\Delta)\}, & (26)\\ \tilde{g}_{II}^{-1}(\bar{y}_2)\ \tilde{G}_I^\Delta(\bar{y}_1) &= \{R_{I,II}^{-1}(\bar{y}_1-\bar{y}_2-\Delta)\ \tilde{G}_I^\Delta(\bar{y}_1)\ R_{I,II}(\bar{y}_1-\bar{y}_2+\Delta)\}\ \tilde{g}_{II}^{-1}(\bar{y}_2). & (27) \end{aligned}$$

Next we express the semi-local generator in terms of its annihilation and creation parts following a procedure similar to that used in Ref. [5],

$$\tilde{G}^{\Delta}(\bar{y}) \equiv [G^{\Delta+}(\bar{y})]^{-1} G^{\Delta-}(\bar{y}) \quad , \tag{28}$$

where the operators $G^{\Delta\pm}(\bar{y})$ satisfy the following exchange algebras that guarantee Eqs. (25) to (27),

$$\begin{aligned} R_{I,II}(\bar{y}_1-\bar{y}_2)\; G_I^{\Delta\pm}(\bar{y}_1)\; G_{II}^{\Delta+}(\bar{y}_2) &= G_{II}^{\Delta\pm}(\bar{y}_2)\; G_I^{\Delta\pm}(\bar{y}_1)\; R_{I,II}(\bar{y}_1-\bar{y}_2), &(29)\\ R_{I,II}(\bar{y}_1-\bar{y}_2+\Delta)\; G_I^{\Delta+}(\bar{y}_1)\; G_{II}^{\Delta-}(\bar{y}_2) &= G_{II}^{\Delta-}(\bar{y}_2)\; G_I^{\Delta+}(\bar{y}_1)\; R_{I,II}(\bar{y}_1-\bar{y}_2-\Delta), &(30)\\ g_I(\bar{y})\; G_{II}^{\Delta\pm}(\bar{y}_2) &= G_{II}^{\Delta\pm}(\bar{y}_2)\; g_I(x)\; R_{I,II}(\bar{y}_1-\bar{y}_2\pm\Delta/2). &(31)\end{aligned}$$

Notice that

$$\left[\sum\nolimits_{n=-\infty}^{n=+\infty} \tilde{j}(\bar{y}+n\Delta), G^{\Delta}(\bar{y})\right] = 0, \tag{32}$$

which manifests what we call the $\mathrm{sl}^{\Delta}(n)\otimes Uq^{1/\Delta}[\mathrm{sl}(n)]$ symmetry of the theory. For $\Delta\to\infty$, Eq. (32) becomes $[\tilde{j}(\bar{y}),\tilde{G}]=0$, manifesting the $\widehat{\mathrm{sl}(n)}\otimes Uq[\mathrm{sl}(n)]$ symmetry of the theory. For $\Delta\to 0$, Eq. (32) becomes $[\tilde{Q},\tilde{j}^q(\bar{y})]=0$, where $\tilde{Q}\equiv\int_{-\infty}^{\infty}\tilde{j}(\bar{y})d\bar{y}=\lim_{\Delta\to 0}\sum_{n=-\infty}^{\infty}[\Delta\tilde{j}(\bar{y}+n\Delta)]$ and $\tilde{j}^q(\bar{y})$ is from the coefficient of the Δ-term in the expansion of the right-hand-side of Eq. (24), manifesting the $\mathrm{sl}(n)\otimes U_q^{\infty}[\mathrm{sl}(n)]$ symmetry of the theory.

Quantum-Group Difference Equation for Correlation Functions Defined in the $U_q^{\Delta}[\mathrm{sl}(2)]$*-Vacuum*

Using Eqs. (28) and (10), we can obtain from Eq. (24)

$$\tilde{g}(\bar{y}+\Delta)=\tilde{g}(\bar{y}-\Delta)\tilde{G}^{\Delta}(\bar{y})=\tilde{g}(\bar{y}-\Delta)[\tilde{G}^{\Delta+}(\bar{y})]^{-1}G^{\Delta-}(\bar{y}). \tag{33}$$

Since we are interested in the vacuum expectation values of the products of the $\tilde{g}$ fields in the $U_q^{\Delta}[\mathrm{sl}(2)]$-vacuum state $\mid 0_q\rangle$ defined by

$$G^{\Delta-}(\bar{y})\mid 0_q\rangle \;=\; \mid 0_q\rangle\;, \quad \text{and} \quad \langle 0_q\mid G^{\Delta+}(\bar{y}) \;=\; \langle 0_q\mid\;. \tag{34}$$

we want to move $\left[G^{\Delta+}(\bar{y})\right]^{-1}$ to the left of $\tilde{g}(\bar{y}-\Delta)$ in Eq. (33). To achieve that feat we use Eq. (31), many matrix relations, and finally reach the goal

$$\tilde{g}(\bar{y}+\Delta)=\left(\left(\left(G^{\Delta+}(\bar{y})\right)^{-1}\right)^{T}\Upsilon\;\tilde{g}^{T}(\bar{y}-\Delta)\right)^{T}G^{\Delta-}(\bar{y}), \tag{35}$$

where the superscript T means matrix transposition (the order of the operators stays put); $\Upsilon \equiv \frac{q+q^{-1}}{q^2+q^{-2}} \times diag(q, q^{-1})$ resulted from

$$\Upsilon_I = (Tr)_{II} \left(P_{I,II} \left(\left((R_{I,II}(0))^{T_I} \right)^{-1} \right)^{T_{II}} \right) \ , \tag{36}$$

where the superscripts T_I and T_{II} indicate transpose of matrices in the tensor spaces I and II respectively.

Using Eq. (34) and Eq. (35), we obtain the quantum-group difference equation for the correlation functions

$$\langle 0_q \mid \tilde{g}_I(\bar{y}_1) \cdots \tilde{g}_L(\bar{y}_l + 2\Delta) \cdots \tilde{g}_N(\bar{y}_n) \mid 0_q \rangle = \langle 0_q \mid \tilde{g}_I(\bar{y}_1) \cdots \tilde{g}_L(\bar{y}_l) \cdots \tilde{g}_N(\bar{y}_n) \mid 0_q \rangle$$

$$\times R_{L,L-1}(\bar{y}_l - \bar{y}_{l-1}) \cdots R_{L,I}(\bar{y}_l - \bar{y}_1) \Upsilon_L R_{L,N}(\bar{y}_l - \bar{y}_n + 2\Delta) \cdots R_{L,L+1}(\bar{y}_l - \bar{y}_{l+1} + 2\Delta). \tag{37}$$

For the special case of "$+2\Delta$" being with $\bar{y}_n$ on the left side of Eq. (37), Eq. (37) simplifies to the following cyclic relation

$$\langle 0_q \mid \tilde{g}_I(\bar{y}_1) \cdots \tilde{g}_N(\bar{y}_n + 2\Delta) \mid 0_q \rangle = \langle 0_q \mid \tilde{g}_N(\bar{y}_n) \tilde{g}_I(\bar{y}_1) \cdots \tilde{g}_{N-1}(\bar{y}_{n-1}) \mid 0_q \rangle \Upsilon_N \ . \tag{38}$$

Similarly, difference equations for products involving the $\tilde{g}^{-1}$'s can also be obtained.

For the two point function, Eq. (37) becomes

$$\langle 0_q \mid \tilde{g}_I(\bar{y}_1) \tilde{g}_{II}(\bar{y}_2 + 2\Delta) \mid 0_q \rangle = \langle 0_q \mid \tilde{g}_I(\bar{y}_1) \tilde{g}_{II}(y_2) \mid 0_q \rangle R_{II,I}(\bar{y}_2 - \bar{y}_1) \Upsilon_{II} \ . \tag{39}$$

Multiplying Eq. (39) from the right by $\mathcal{P}^q_{j_{12}=0}$ and using the fact $\langle 0_q \mid \tilde{g}_I \tilde{g}_{II} \mid 0_q \rangle \mathcal{P}^q_{j_{12}=0}$ $= \langle 0_q \mid \tilde{g}_I \tilde{g}_{II} \mid 0_q \rangle$, which can be shown using the definition of $\mid 0_q \rangle$ given by Eq. (34), we obtain

$$\langle 0_q \mid \tilde{g}_I(\bar{y}_1) \tilde{g}_{II}(\bar{y}_2 + 2\Delta) \mid 0_q \rangle = \langle 0_q \mid \tilde{g}_I(\bar{y}_1) \tilde{g}_{II}(\bar{y}_2) \mid 0_q \rangle \ q^{-\Delta_0 \varepsilon(\bar{y}_1 - \bar{y}_2)} \frac{q + q^{-1}}{q^2 + q^{-2}}, \tag{40}$$

where the last factor on the right is from $R_{I,II}(\bar{y}_1 - \bar{y}_2) \Upsilon_{II} \mathcal{P}^q_{j_{12}=0} = \mathcal{P}^q_{j_{12}=0} q^{-\Delta_0 \varepsilon(\bar{y}_1 - \bar{y}_2)}(b/a)$ with $b/a \equiv (q + q^{-1})/(q^2 + q^{-2}) = ([2]_q)^2/[4]_q$ and the fact that $\mathcal{P}^q_{j_{12}=0}$ multiplying the vacuum expectation value becomes unit.

We find the solution to Eq. (40). It can be written in the following form

$$\langle 0_q \mid \tilde{g}_I(\bar{y}_1)\tilde{g}_{II}(\bar{y}_2) \mid 0_q\rangle$$

$$= A_0 Exp\Big\{-\Big(\frac{\bar{y}_1-\bar{y}_2}{2\Delta}\Big)ln\Big(\frac{q+q^{-1}}{q^2+q^{-2}}\Big) + \Big[\Big(\frac{\bar{y}_1-\bar{y}_2}{2\Delta}\Big) + 2\sum_{n=1}^{\infty}\theta\Big(-\frac{\bar{y}_1-\bar{y}_2}{2\Delta} - n\Big)\Big] ln(q^{\Delta_0})\Big\}, \tag{41}$$

where A_0 is an arbitrary constant; $\theta(x) = 0, \frac{1}{2}, 1$ for $x < 0, x = 0, x > 0$, respectively. This expression for the solution is continuous in the $\bar{y}_1 - \bar{y}_2 > 0$ region. For expressing the solution in a function that is continuous in the $\bar{y}_1 - \bar{y}_2 < 0$ region, we replace $(\bar{y}_1 - \bar{y}_2)$ by $-(\bar{y}_1 - \bar{y}_2)$ and $\sum_{n=1}^{\infty}$ by $\sum_{n=0}^{\infty}$ in the square bracket of the above equations.

Concluding remarks

The group-valued local quantum fields $\tilde{g}$ and $\tilde{g}^{-1}$ and their exchange algebras form the foundation of the quantum WZNW field theory. Understanding the meaning of the spatial dependence of the R matrix of the exchange algebras is essential in our formulation of the theory. Being bimodule quantum fields and having quantum-group structures are generic features of non-Abelian group-valued local quantum fields. From these group-valued quantum fields, the content of the theory in Lie-algebra-valued fields can easily be derived. The other way around is much harder. The clear exposition of the bimodule properties of group-valued fields in our formulation leads us to the explicit construction of the quantum-group generators, the $U_q^{\Delta}[\mathrm{sl}(n)]$-vacuum, and the derivation of the quantum-group difference equations for the correlation functions defined in the $U_q^{\Delta}[\mathrm{sl}(n)]$-vacuum. This way of understanding the quantum WZNW theory has served us well in developing the quantum Self-dual Yang-Mills theory, which is a quantum field theory with interactions in four dimensions.

Quantum states of non-abelian group-valued local fields may well exist in nature. It is important to look for them. If they do exist, we would like to call them bimodulons [25].

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